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LETTER TO THE EDITOR

# A method for accurate stability bounds in a small denominator problem

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**Abstract.** We consider the problem of obtaining realistic lower bounds for the Siegel radius. Recent advances of the analysis of Siegel disks allow us to give a very accurate numerical algorithm based on rigorous results.

We find that for non-quadratic polynomial maps the maximal Siegel radius might correspond to rotation numbers different from the golden mean.

Small denominators are the most important difficulty in the study of classical perturbation theory. One of the basic unsolved problems is to find methods for obtaining accurate estimates of the breakdown threshold of the invariant manifolds. These invariant manifolds, usually circles or tori, are preserved under small perturbations of integrable systems. A rigorous and computationally effective method is not yet available. The best rigorous results, due to computer-assisted KAM proofs (Celletti and Chierchia 1987, De La Llave and Rana 1986, Liverani *et al* 1984, Liverani and Turchetti 1986), give lower bounds which differ from the values resulting from numerical studies by less than 10%.

We report here briefly on a simple numerical algorithm, based on theorems due to Herman (1985, 1987a, b), which permits us to compute very accurate estimates for the simplest small denominator problem, namely the Siegel centre problem. We refer to Marmi (1988b) for more details of the proofs and for some applications to polynomial maps of $\boldsymbol{C}$.

Suppose that $f:\boldsymbol{C}\to\boldsymbol{C}$ is analytic and has a fixed point at $z=0$ with eigenvalue $\lambda := f'(0)=\exp(2\pi i\omega)$, so that $f(z)=\lambda x+\Sigma_{k=2}^{+\infty} f_k z^k$. Assume that $\omega$ is irrational and verifies a Diophantine condition, i.e. there exist two positive constants $\gamma$ and $\mu$ such that for all $p, q\in\boldsymbol{Z}$, $q\neq 0$, $|\omega-p/q|\geqslant\gamma q^{-\mu}$. Then Siegel (1942) proved that there exists a unique analytic diffeomorphism $\Phi$, from a disk $D_r := \{w\in\boldsymbol{C}\,|\,|w|<r\}$ to a neighbourhood of $z=0$, such that $\Phi(0)=0$, $\Phi'(0)=1$ and

$$(f\circ\Phi)(w)=\Phi(\lambda w) \tag{1}$$

i.e. $f$ is analytically conjugated to its linear part on a neighbourhood $\Phi(D_r)$ of its fixed point. The maximal open connected neighbourhood $U$ of $z=0$ which has this property and is invariant under $f$ is called the Siegel singular domain (Siegel disk). It is foliated into invariant manifolds analytically equivalent to circles with rotation number $\omega$. The

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Siegel radius $r_S$ is the radius of the disk $D_{r_S} = \Phi^{-1}(U)$, i.e. the radius of convergence of the power series expansion $\Sigma_{k\geq 1} \Phi_k w^k$ of $\Phi(w)$. The coefficients $\Phi_k$ can be easily computed by matching powers in (1): clearly $\Phi_1 = 1$ and for $k \geq 2$ one has

$$\Phi_k = (\lambda^k - \lambda)^{-1} \sum_{m=2}^{k} f_m \sum_{k_1+\ldots+k_m=m} \Phi_{k_1} \ldots \Phi_{k_m} \tag{2}$$

where $(\lambda^k - \lambda)$ is the small denominator and in the sum each $k_j \geq 1$.

Upper bounds to the Siegel radius can be obtained from the knowledge of the coefficients $\Phi_k$, applying the Hadamard criterion (see Liverani *et al* 1984). Moreover, by the Bieberbach–de Branges theorem (De Branges 1985), for all $k \geq 1$ one has

$$|\Phi_k| r_S^{k-1} \leq k \tag{3}$$

and by the area formula for univalent functions (Pommerenke 1975)

$$\text{area}\,(\Phi(D_{r_S})) = \text{area}\,(U) = \pi \sum_{k=1}^{\infty} k|\Phi_k|^2 r_S^{2k}. \tag{4}$$

The area of the Siegel domain $U$ can be numerically computed from the knowledge of the trajectory of a critical point of $f$ (i.e. a point $z_0$ where $f'(z_0) = 0$) given by a finite number of iterations of the map. In fact, no critical points of $f$ can be contained in $U$ because $f_U$ is injective. From the classical theory of Fatou and Julia (see Blanchard 1984) one knows that $\partial U$ is contained in the closure of the forward orbits $\{f^n(z_0) \mid n \geq 0\}$ of the critical points. Indeed Herman (1985) has proved that, if $U$ has compact closure, $f|_{\partial U}$ is injective and $\omega$ is Diophantine, then there is a critical point of $f$ on $\partial U$. In particular he showed that this is the case for mappings of the form

$$f(z) = \exp(2\pi i \omega) z + z^n \tag{5}$$

with $n \geq 2$.

Analytical proofs of the Siegel theorem (De La Llave 1983, Marmi 1988a, Siegel 1942) give rigorous lower bounds which are unfortunately far from being realistic. Computer-assisted proofs (De La Llave and Rana 1986, Liverani *et al* 1984, Liverani and Turchetti 1986) help to improve these estimates so that lower and upper bounds differ by less than 20%. However, in this case a considerable amount of numerical work and of computing time is required (about one hour on a VAX 11/750 for one single estimate (De La Llave and Rana 1986)).

Much more accurate lower bounds for the Siegel radius can be obtained by applying the following remark (Herman 1987a, Marmi 1988a). From (1) one clearly has $(f^j \circ \Phi)(w) = \Phi(\lambda^j w)$ for all $w \in D_{r_S}$ and $j \geq 0$. Thus for all $m \geq 1$

$$\frac{1}{m} \sum_{j=0}^{m-1} \log |f^j(z)| = \frac{1}{m} \sum_{j=0}^{m-1} \log |\Phi(\lambda^j w)|$$

where $z = \Phi(w) \in U$. Notice that $\log|\Phi(w)|$ is harmonic and as $\Phi$ is an analytic diffeomorphism of $D_{r_S}$ onto $U$, with $\Phi(0) = 0$, it has neither poles nor zeros but $w = 0$, so that $\int_0^1 \log|\Phi(r \exp(2\pi i\theta))|\, d\theta = \log r$ for all $r < r_S$. Note in addition that $w \to \lambda w$ is uniquely ergodic on the circle $|w| = r < r_S$. Therefore, applying the ergodic theorem, one has that for all $z \in U$

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log|f^j(z)| = \log r. \tag{6}$$

Taking the radial limit $r \to r_S$ one also shows that (6) holds when $r = r_S$ for almost everywhere $z \in \partial U$ with respect to the harmonic measure.

When $r < r_S$ it is also possible to give a rigorous estimate of the speed of convergence in (6). Indeed, if $[a_0, a_1, a_2, \ldots]$ denotes the continued fraction expansion of $\omega$, and $(p_k/q_k)_{k \geq 0}$ its partial fractions, then

$$(a_k+2)^{-1} q_k^{-2} \leq |\omega - p_k/q_k| \leq a_{k+1}^{-1} q_k^{-2}$$

the successive closest recurrences of an orbit are given by $f^{q_k}(z)$, and one can show that, for all $z \in U$,

$$\left| \frac{1}{q_k} \sum_{j=0}^{q_k-1} \log|f^j(z)| - \log r \right| \leq \frac{\operatorname{var}(\log|\Phi|)}{q_k}$$

where var denotes the variation on $|w| = r$.

In a recent survey article (Douady 1987) it has been conjectured that, as numerical experiments suggest (Mackay and Percival 1987, Manton and Nauenberg 1983, Widom 1983), for quadratic mappings the boundary of the Siegel singular domain is a quasicircle, i.e. the image of the unit circle under a quasiconformal map. We recall that a quasicircle $J \subsetneq \boldsymbol{C}$ is characterised by the fact that there exists a positive constant $K$ such that for all $z_1, z_2 \in J$

$$\min(\operatorname{diam} J_1, \operatorname{diam} J_2) \leq K|z_1 - z_2|$$

where $J_1$ and $J_2$ are the two arcs of which $J \setminus \{z_1, z_2\}$ consists. For instance, a quasicircle may have (non-zero-angle) corners but may not have (zero-angle) cusps.

Actually Herman has recently proven (Herman 1987b) that, if $f(z) = \exp(2\pi i\omega)z + z^2$ and $\omega$ is Diophantine with exponent $\mu = 2$, then $\partial U$ is a quasicircle. In this case we can prove that there exists a number $\chi \in [0, 1[$ such that, for all $z \in \partial U$,

$$\left| \frac{1}{q_k} \sum_{j=0}^{q_k-1} \log|f^j(z)| - \log r \right| \leq \frac{8}{r_S}\left(\frac{2\pi}{q_k}\right)^{1-\chi}. \tag{7}$$

In fact, $\chi$ is the norm of the Grunsky operator (Pommerenke 1975) associated with the univalent function $g(x) = r_S/\Phi(r_S/x)$ defined for $|x| > 1$.

In table 1 we have reported the estimates of the Siegel radius resulting from the application of (6) to the orbit $f^{q_k}(z)$ of the point $z = (1-\varepsilon)z_0$, where $z_0 = -\frac{1}{2}\exp(2\pi i\omega)$

**Table 1.** Estimates of the Siegel radius for the map $f(z) = \exp(2\pi i\omega)z + z^2$, obtained applying (6) to the first $q_k$ iterates of $z = (1-\varepsilon)z_0$, where $z_0 = -\frac{1}{2}\exp(2\pi i\omega)$ is the critical point of $f$. For comparison also the etimates obtained applying Hadamard and Bieberbach-de Branges theorems to $\Phi_{q_k}$ and the area theorem are reported. $\omega = (\sqrt{5}+1)/2$.

| | $q_k$ | | | |
|---|---|---|---|---|
| $\varepsilon$ | 377 | 610 | 987 | 1597 |
| $10^{-1}$ | 0.316 637 | 0.316 642 | 0.316 642 | 0.316 642 |
| $10^{-2}$ | 0.324 704 | 0.324 839 | 0.324 918 | 0.324 963 |
| $10^{-3}$ | 0.324 820 | 0.324 964 | 0.325 057 | 0.325 114 |
| $10^{-4}$ | 0.324 822 | 0.324 966 | 0.325 058 | 0.325 116 |
| $10^{-5}$ | 0.324 822 | 0.324 966 | 0.325 058 | 0.325 116 |
| 0 | 0.324 822 | 0.324 966 | 0.325 058 | 0.325 116 |
| Hadamard | 0.332 66 | 0.330 06 | 0.328 36 | 0.327 25 |
| Bieberbach-de Branges | 0.336 96 | 0.332 95 | 0.330 29 | 0.328 54 |
| Area criterion $r_S \leq 0.350$ | | | | |

is the critical point of the quadratic map $f(z)=\exp(2\pi i\omega)z+z^2$. The values of $\varepsilon$ range from $10^{-1}$ to 0, and $\omega$ is equal to $(\sqrt{5}+1)/2=[1,1,\ldots]$. For comparison upper bounds by Hadamard, Bieberbach-de Branges and area theorems are also given. It is clear that the application of (6) gives the best estimates and converges pretty fast.

For this map we have also computed the Siegel radius from $q_k=317\,811$ iterations of the critical point $z_0$ (corresponding to the term $k=27$ of the Fibonacci sequence). The result is $r_S^*=0.325\,210\,835\,69$, and an analysis of the numerical stability (the round-off error being smaller than $10^{-15}$) and of the speed of convergence of (6), suggests that the first six digits should be exact. The computer time needed is approximately 5 min on a PC.

In figure 1 we have plotted $\log|(1/q_k)\sum_{j=0}^{q_k-1}\log|f^j(z)|-\log r_S^*|$ as a function of $\log q_k$. As one should expect from the error estimate (7) the points lie on a straight line; the slope is $\alpha=0.99$.

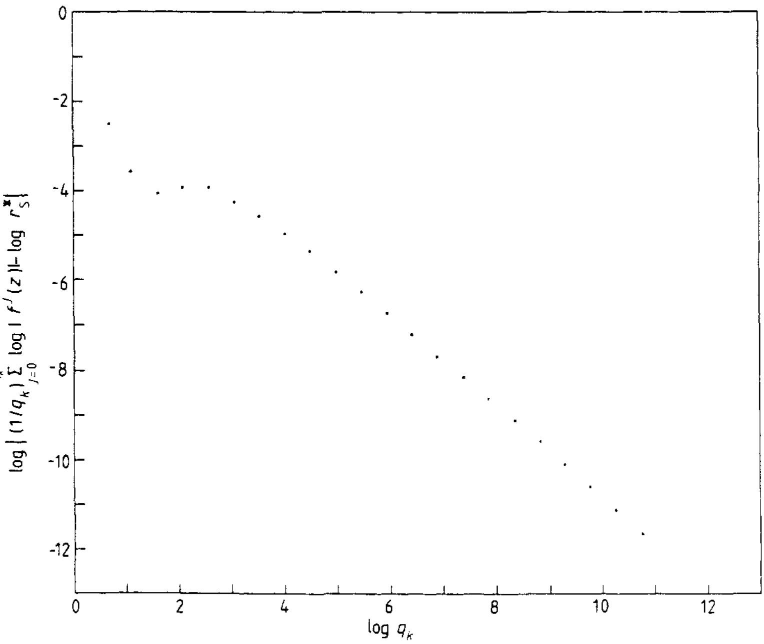


**Figure 1.** $\log|(1/q_k)\sum_{j=0}^{q_k-1}\log|f^j(z)|-\log r_S^*|$ plotted as a function of $\log q_k$.

These and other tests provide a good evidence of the reliability and accuracy of (6) for obtaining realistic estimates of the Siegel radius $r_S$. Therefore we use it in Marmi (1988b) in order to study the dependence of $r_S$ on the degree $n$ of the maps (5) and on the continued fraction of the rotation number $\omega$. Table 2 and figure 2 report some preliminary results.

It is of some interest to remark that for non-quadratic mappings the maximum Siegel radius does not correspond to the golden mean rotation number $\omega=(\sqrt{5}+1)/2$, Indeed, if one considers rotation numbers with a constant continued fraction expansion $\omega=[p,p,\ldots]$ for maps (5) of degree $n\geqslant 3$, then $r_S$ is not a monotonic function of $p$. This result is not too surprising if one takes into account that in this case the power series of $\Phi$ has the structure $\Phi(w)=w\tilde{\Phi}(w^{n-1})$, i.e. it is a power series in $w^{n-1}$. Symmetries like the previous one might also appear in the applications of Hamiltonian

**Table 2.** Estimates of the Siegel radius for the maps $f(z) = \exp(2\pi i\omega)z + z^n$, where $\omega = [p, p, \ldots] = \frac{1}{2}[(p^2+4) + p]$, $p = 1, \ldots, 10$ and $n = 2, \ldots, 5$. The first $q_k(p)$ iterates of the critical point, with $q_k(p) \leq 10\,000$ have been used. The first four digits are significant.

| | $n$ | | | |
|---|---|---|---|---|
| $p$ | 2 | 3 | 4 | 5 |
| 1 | 0.325 17 | 0.437 79 | 0.498 96 | 0.560 33 |
| 2 | 0.324 09 | 0.426 18 | 0.509 02 | 0.559 76 |
| 3 | 0.320 61 | 0.438 59 | 0.469 79 | 0.570 72 |
| 4 | 0.314 99 | 0.419 50 | 0.518 32 | 0.494 61 |
| 5 | 0.307 82 | 0.440 32 | 0.521 06 | 0.573 25 |
| 6 | 0.299 64 | 0.413 76 | 0.465 90 | 0.570 05 |
| 7 | 0.290 96 | 0.438 60 | 0.522 68 | 0.576 21 |
| 8 | 0.282 00 | 0.408 77 | 0.522 80 | 0.501 71 |
| 9 | 0.273 00 | 0.434 19 | 0.467 62 | 0.576 44 |
| 10 | 0.264 14 | 0.403 72 | 0.521 73 | 0.569 49 |

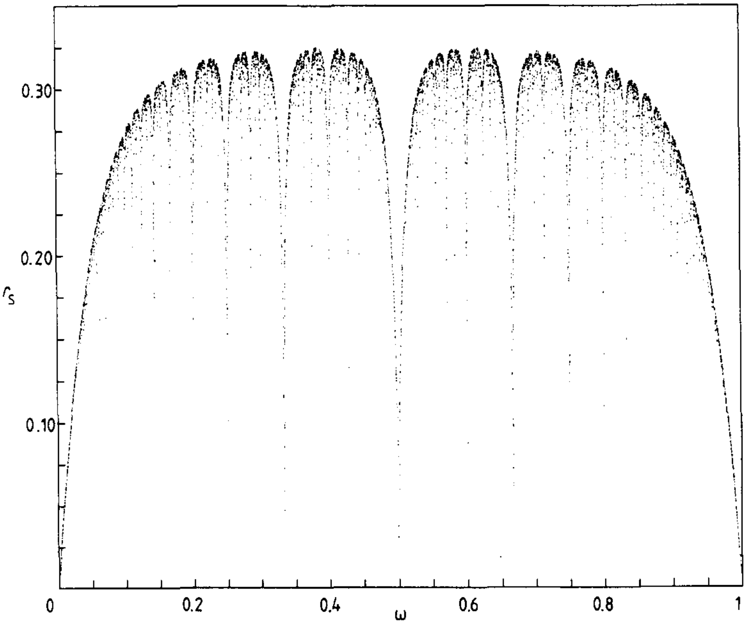


**Figure 2.** The Siegel radius as a function of $\omega \in [0, 1]$ for the quadratic map, computed by means of 20 000 iterates of the critical point at 7000 uniformly distibuted random rotation numbers $\omega$. The numerical error is approximately $10^{-4}$.

perturbation theory to some special systems. We expect that, in these cases, similar results are possible and therefore one should try to obtain realistic bounds in small denominator problems for 'classes' of frequencies (like the 'noble' numbers (Percival 1982)) or, better, for positive measure sets.

Yoccoz (1985) has recently proved a remarkable result in this direction: for the quadratic map the set $Y \subseteq S^1$ of rotation numbers $\omega$ which have a Siegel radius $r_S(\omega) \geq \frac{1}{4}$ has positive Lebesgue measure. In figure 2 we exhibit the Siegel radius $r_S$ at 7000 uniformly distributed random rotation numbers $\omega \in [0, 1]$. The self-similarity of the figure is evident; notice also the analogy with figure 2 of Percival's study of the semistandard map with 'noble' frequencies. From our data we can also obtain an estimate of the measure of the set $Y$ of Yoccoz's theorem; we find meas $Y \geqslant 0.74$.

To conclude we want to stress that it would be very interesting to extend Yoccoz's result to non-quadratic complex mappings, especially to area-preserving or Hamiltonian systems. Some preliminary numerical results on the polynomial maps (5) indicate that this should be possible.

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## References

Blanchard P 1984 *Bull. Am. Math. Soc.* **11** 85–141
Celletti A and Chierchia L 1987 *Construction of analytic KAM surfaces and effective stability bounds. Preprint* ETH, Zürich
De Branges L 1985 *Acta Math.* **154** 137–52
De La Llave R 1983 *J. Math. Phys.* **24** 2118–21
De La Llave R and Rana D 1986 *Proof of accurate bounds in small denominators problems. Preprint* Princeton University
Douady A 1987 *Astérisque* **152–3** 151–72
Herman M R 1985 *Commun. Math. Phys.* **99** 593–612
—— 1987a *Proc. 8th Int. Congr. on Mathematical Physics, Marseille, 1986* ed M Mebkhout and R Sénéor (Singapore: World Scientific)
—— 1987 unpublished
Liverani C, Servizi G and Turchetti G 1984 *Lett. Nuovo Cimento* **39** 417
Liverani C and Turchetti G 1986 *J. Stat. Phys.* **45** 1071–86
Mackay R S and Percival I C 1987 *Physica* **26D** 193–202
Manton N S and Nauenberg M 1983 *Commun. Math. Phys.* **89** 555–70
Marmi S 1988a *Meccanica* to be published
—— 1988b in preparation
Percival I C 1982 *Physica* **6D** 67–77
Pommerenke C 1975 *Univalent Functions* (Göttingen: Vandenhoeck and Ruprecht)
Siegel C L 1942 *Ann. Math.* **43** 807–12
Widom M 1983 *Commun. Math. Phys.* **92** 121–36
Yoccoz J C 1985 *Siegel's theorem for quadratic polynomials. Preprint*